- **Solve the equation: $\sqrt[3]{x^3 - 3x^2} + 1 = x$**

(Note that the "plus one" is outside the cube root.)

Since this is a cube root, I'll cube both sides to undo the radical. But first, I want to isolate the radical:

$$\begin{aligned}
\sqrt[3]{x^3 - 3x^2} + 1 &= x \\
\sqrt[3]{x^3 - 3x^2} &= x - 1 \\
\left(\sqrt[3]{x^3 - 3x^2}\right)^3 &= (x-1)^3 \\
x^3 - 3x^2 &= (x-1)(x-1)(x-1) \\
x^3 - 3x^2 &= x^3 - 3x^2 + 3x - 1 \\
0 &= 3x - 1 \\
\tfrac{1}{3} &= x
\end{aligned}$$

Remember to check the solution:

$$\begin{aligned}
\sqrt[3]{x^3 - 3x^2} + 1 &= x \\
\sqrt[3]{\left(\tfrac{1}{3}\right)^3 - 3\left(\tfrac{1}{3}\right)^2} + 1 &\stackrel{?}{=} \tfrac{1}{3} \\
\sqrt[3]{\tfrac{1}{27} - 3\left(\tfrac{1}{9}\right)} + 1 &\stackrel{?}{=} \tfrac{1}{3} \\
\sqrt[3]{\tfrac{1}{27} - \tfrac{1}{3}} + 1 &\stackrel{?}{=} \tfrac{1}{3} \\
\sqrt[3]{-\tfrac{8}{27}} + 1 &\stackrel{?}{=} \tfrac{1}{3} \\
-\tfrac{2}{3} + 1 &\stackrel{?}{=} \tfrac{1}{3} \\
\tfrac{1}{3} &= \tfrac{1}{3} \quad \text{...yes!}
\end{aligned}$$

So the solution is $x = {}^{1}/_{3}$.

- **Solve the equation:** $x+1=\sqrt[4]{x^4+4x^3-x}$

Since this is a fourth root, I'll raise both sides to the fourth power:

$$
\begin{aligned}
x+1 &= \sqrt[4]{x^4+4x^3-x} \\
(x+1)^4 &= \left(\sqrt[4]{x^4+4x^3-x}\right)^4 \\
x^4+4x^3+6x^2+4x+1 &= x^4+4x^3-x \\
6x^2+4x+1 &= -x \\
6x^2+5x+1 &= 0 \\
(2x+1)(3x+1) &= 0 \\
x &= -\tfrac{1}{2} \\
x &= -\tfrac{1}{3}
\end{aligned}
$$

Then I'll check my answers:

$x = {}^{-1}/_{2}$:

$$
\begin{aligned}
x+1 &= \sqrt[4]{x^4+4x^3-x} \\
\left(-\tfrac{1}{2}\right)+1 &\stackrel{?}{=} \sqrt[4]{\left(-\tfrac{1}{2}\right)^4+4\left(-\tfrac{1}{2}\right)^3-\left(-\tfrac{1}{2}\right)} \\
\tfrac{1}{2} &\stackrel{?}{=} \sqrt[4]{\tfrac{1}{16}-\tfrac{4}{8}+\tfrac{1}{2}} \\
\tfrac{1}{2} &\stackrel{?}{=} \sqrt[4]{\tfrac{1}{16}} \\
\tfrac{1}{2} &= \tfrac{1}{2}
\end{aligned}
$$

- **Solve $| x - 3 | = | 3x + 2 | - 1$**

  First I need to find the break-points for each of these absolute values. Where do the arguments (the expressions between the bars) of these absolute values switch from being positive inside the bars to being negative? I'll look at each absolute value separately.

  $| x - 3 | \geq 0$ for $x \geq 3$

  $| 3x + 2 | \geq 0$ for $x \geq -2/3$

How did I arrive at these conclusions? I know that $x - 3 = 0$ at $x = 3$, and that the line $y = x - 3$ has a positive slope and thus an increasing line. So $x - 3 = 0$ at $x = 3$, and $x - 3$ must be positive after $x = 3$. Also, I know that $3x + 2 = 0$ at $x = -2/3$, and that the line $y = 3x + 2$ has a positive slope and thus an increasing line. So $3x + 2 = 0$ at $x = -2/3$, and $3x + 2$ must be positive after $x = -2/3$.

These points, $x = -2/3$ and $x = 3$, are where the absolute-value expressions equal zero. Since these expressions must be negative or positive for other x-values, then these points divide the number line into intervals (before $x = -2/3$, between $x = -2/3$ and $x = 3$, and after $x = 3$), each of which should be considered separately.

The zeroes of the two absolute-value expressions give me three intervals: $(-\textit{infinity}, -2/3)$, $(-2/3, 3)$, and $(3, \textit{infinity})$. On the first interval, both absolute-value expressions will have negative values, so I'll need to change the signs on both of them when I take the bars off.

$| x - 3 | = | 3x + 2 | - 1$

$-(x - 3) = -(3x + 2) - 1$

$-x + 3 = -3x - 2 - 1$

$2x = -6$

$x = -3$

This tells me that the solution to the original equation, *on the interval (–infinity, –2/3)*, is $x = -3$. Since $x = -3$ is contained within this interval, this solution is valid.

On the second interval (where x is between $-2/3$ and 3), the absolute value on the left-hand side of the equation, $|\, x - 3 \,|$, has a negative argument; I'll have to change its sign when I take off the bars. But the absolute value on the right-hand side of the equation, $|\, 3x + 2 \,|$, has a positive argument, so I can just take the bars off.

$$| x - 3 | = | 3x + 2 | - 1$$

$$-(x - 3) = 3x + 2 - 1$$

$$-x + 3 = 3x + 1$$

$$2 = 4x$$

$$1/2 = x$$

This tells me that the solution to the original equation, *on the interval (–2/3, 3)*, is $x = 1/2$. Since $1/2$ is between $-2/3$ and 3, this solution is valid.

On the third interval (where x is 3 or more), both absolute values have positive arguments, so I can just take the bars off.

$$| x - 3 | = | 3x + 2 | - 1$$

$$x - 3 = 3x + 2 - 1$$

$$-4 = 2x$$

$$-2 = x$$

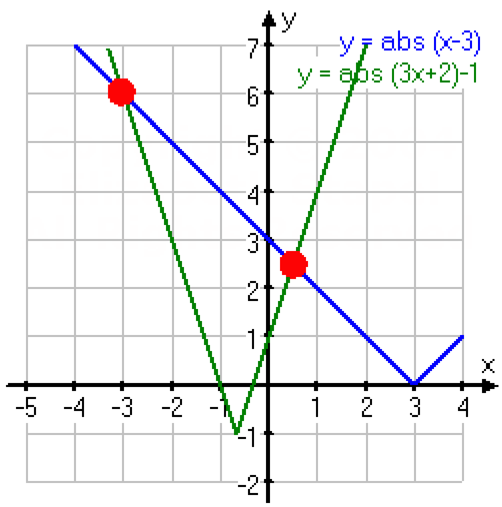
However, on this interval, x was already fixed as being greater than 3; the solution then cannot be "$x = -2$", since -2 is actually less than 3. So "$x = -2$" is not actually a valid solution!

However, on this interval, x was already fixed as being greater than 3; the solution then cannot be "$x = -2$", since -2 is actually less than 3. So "$x = -2$" is not actually a valid solution!

Then the answer is:

$x = -3$ or $x = {}^1/_2$

To confirm this graphically, look for the intersections of $y_1 = | x - 3 |$ and $y_2 = | 3x + 2 | - 1$:



You don't often need to take different intervals into consideration— but sometimes you do. So make sure you understand the last exercise above.